Section 5.4: Mechanical vibrations

Vocabulary:

- simple harmonic motion
- Amplitude, frequency, phase angle
- Damping: underdamped, overdamped, critical

We learn:

- Hooke's law for springs.
- How to model s.h.m. with a differential equation, including damping.
- How to express s.h.m. with a single cosine function

We don't need to know:

- circular frequency (rather than frequency)
- time lag (bottom of page 305
- formulas for frequency and period (p. 304-5)
- pseudofrequency, pseudoperiod
- time-varying amplitude
- the derivation of s.h.m. as an approximation for the motion of a pendulum

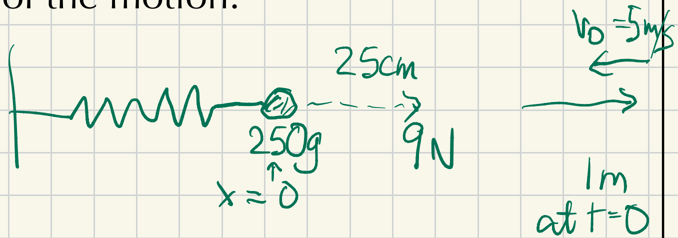
Page 311 question 4.

A body of mass 250g is attached to the end of a spring that is stretched by 25cm by a force of 9N. At $t = 0$ the body is pulled 1m to the right and set in motion with $v_0 = 5$m/s to the left.

a. Find the position $x(t)$ of the body at time t in the form $C\cos(\omega_0 t - \alpha)$

b. Find the amplitude and period (and frequency) of the motion.

Solution



Hooke's law: the force a spring pulls with is proportional to displacement.

Thus if 9N are needed to pull it .25m then $4 \cdot 9N = 36N$ are needed to pull it 1 m.

Force $F = -kx$. Here $k = 36$ N/m

Differential equation.

$$\text{Force} = \boxed{mx'' = -kx}$$

Here $0.25x'' + 36x = 0$, $x'' + 144x = 0$

Char. eqn. $r^2 + 144 = 0$, $r = \pm 12i$

General solution: $x = A\cos(12t) + B\sin(12t)$

Find A and B.

When $t = 0$ $x_0 = x(0) = 1 = A$

$x'(t) = -12A\sin 12t + 12B\cos 12t$

$x'(0) = v_0 = -5 = 12B$ $B = -5/12$

Solution: $x = \cos 12t - \frac{5}{12}\sin 12t$.

Solution: $x(t) = \cos(12t) - (5/12)\sin(12t)$

The period is the time taken to complete one cycle
at $12t = 2\pi$, $t = \frac{2\pi}{12}$

The frequency is the number of cycles per time interval
$= 1/\text{period} = 12/2\pi$

We write the solution in the form

$$C\cos(\omega_0 t - \alpha)$$ $\omega_0/2\pi$ is the frequency

α is called the phase angle.

Use the identity $\cos(A+B) = \cos A\cos B - \sin A \sin B$

See next page for a comment

Take $B = 12t$. Choose C and A so that $C\cos A = 1$ and $C\sin A = 5/12$

To do this, note: $(C\cos A)^2 + (C\sin A)^2$

$$= C^2 = 1^2 + \left(\frac{5}{12}\right)^2 = \frac{12^2+5^2}{12^2} = \frac{13^2}{12^2}$$

$C = \frac{13}{12}$ is the amplitude.

$x'(t) = -13\sin(12t-\alpha)$, $x'(0) = -13\sin(-\alpha) < 0$

Also $\frac{C\sin A}{C\cos A} = \frac{5/12}{1} = \frac{5}{12} = \tan A$

$$A = \tan^{-1}\left(\frac{5}{12}\right)$$

$$x(t) = C(\cos A\cos 12t - \sin A \sin 12t)$$

$$= \frac{13}{12}\cos(A + 12t) = \frac{13}{12}\cos(12t - \alpha)$$

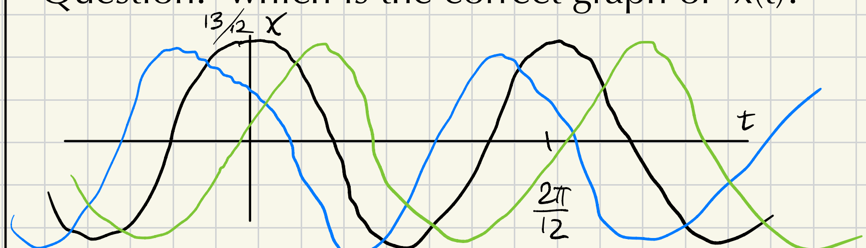
where $\alpha = -\tan^{-1} 5/12$.

Question: which quadrant does the phase angle lie in?

a. 1 b. 2 c. 3 d. 4 ✓

2	1
3	4

Question: which is the correct graph of x(t)?



A black ✓B blue C green

Identify blue at left using derivative

In deriving how to write
x(t) = cos(12t) - (5/12) sin(12t) in the form
C cos (omega t - alpha) it would have been better to use the identity
cos(A-B) = cos A cos B + sin A sin B because this deals with the minus signs better.

Let A = 12t here and B = alpha. We try to find C and B so that C cos B = 1 and C sin B = -5/12

We get

$$C^2 = C^2 \cos^2 A + C^2 \cos^2 B$$

$$= 1 + \left(-5/12\right)^2 = \left(\frac{13}{12}\right)^2$$

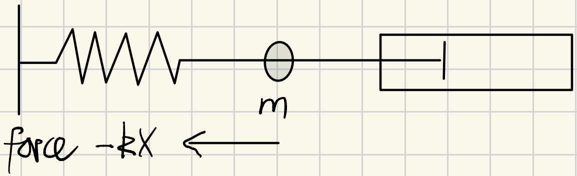
so $C = \dfrac{13}{12}$

and $\tan B = \dfrac{C \sin B}{C \cos B} = \dfrac{-5/12}{1} = -\dfrac{5}{12}$

Thus $x(t) = \dfrac{13}{12} \cos\left(12t - \tan^{-1}\left(-\dfrac{5}{12}\right)\right)$

Page 312 question 19.
A mass m is attached to both a spring (with spring constant k) and a dash pot (with damping constant c). The mass has initial position x_0, initial velocity v_0. Find the position function x(t) and determine whether the motion is overdamped, critically damped, or underdamped.

m=4, c=20, k=169, x_0=4, v_0=16



Equation Force $= mx'' = -kx - cx'$

Here $4x'' + 20x' + 169 = 0$

$$4r^2 + 20r + 169 = 0$$

roots $r = -\frac{5}{2} \pm 6i$

$$x(t) = e^{-5/2\,t}(A\cos 6t + B\sin 6t).$$

$x(0) = A = 4$, $x'(t) = -\frac{5}{2}e^{-5/2\,t}(A\cos 6t + B\sin 6t) + e^{-5/2\,t}(-6A\sin 6t + 6B\cos 6t)$

$x'(0) = -\frac{5}{2}A + 6B = 16$, $B = 13/3$

$$x(t) = e^{-5/2\,t}\left(4\cos 6t + \frac{13}{3}\sin 6t\right)$$

Without the $e^{-5/2\,t}$ term the amplitude is $C = \sqrt{4^2 + \left(\frac{13}{3}\right)^2} = \frac{\sqrt{313}}{3}$

$$x(t) = e^{-5/2\,t}\,\frac{\sqrt{313}}{3}\cos(6t - \alpha)$$

where $\tan(\alpha) = \frac{13/3}{4} = \frac{13}{12}$.

See next page.

The last question says more than this. In brief:

Page 312 question 19.
A mass m is attached to both a spring (with spring constant k) and a dash pot (with damping constant c). The mass has initial position x_0, initial velocity v_0. Find the position function $x(t)$ and determine whether the motion is overdamped, critically damped, or underdamped.

Write $x(t)$ in the form
$C_1 e^{-pt} \cos(\omega t - \alpha)$.
Also, if we put $c = 0$ find the corresponding function
$u(t) = C_0 \cos(\omega t - \alpha)$.

$m=4$, $c=20$, $k=169$, $x_0=4$, $v_0=16$

We got underdamping.

Our solution was

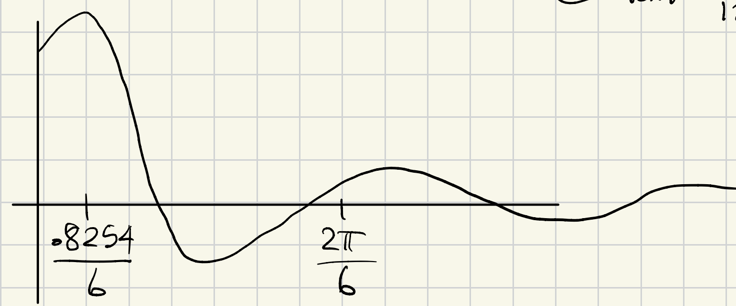
$$x(t) = e^{-\frac{5t}{2}}\left(4\cos 6t + \frac{13}{3}\sin 6t\right)$$

$$\text{Amplitude} = \sqrt{4^2 + \frac{13^2}{3^2}} = \frac{\sqrt{313}}{3} \quad \left(\text{without the } e^{-\frac{5t}{2}}\right)$$

$$x(t) = \frac{\sqrt{313}}{3} e^{-5t/2}\left(\frac{12}{\sqrt{313}}\cos 6t + \frac{13}{\sqrt{313}}\sin 6t\right)$$

$$= \frac{\sqrt{313}}{3} e^{-5t/2} \cos(6t - 0.8254)$$

$$\tan^{-1}\frac{13}{12}$$



Extra material with page 312 question 19:
We just did
m=4, c=20, k=169, x_0=4, v_0=16
With equation
$4x'' + 20x' + 169x = 0$

Next we do the same, but with $k = 16$.
This means the spring is weaker.

$$4x'' + 20x' + 16x = 0$$

$$4r^2 + 20r + 16 = 0$$

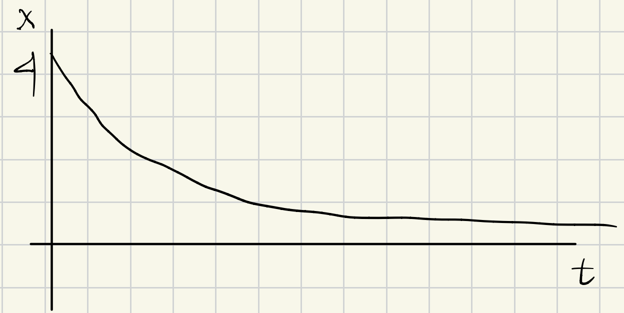
$$4(r^2 + 5r + 4) = 0$$

$$4(r+1)(r+4) = 0. \quad \text{Roots } r = -1, -4$$

General solution $x(t) = Ae^{-t} + Be^{-4t}$

Apply initial conditions:

$$x(t) = 10\tfrac{2}{3}e^{-t} - 6\tfrac{2}{3}e^{-4t}$$



There is no oscillation. The roots are real.
This is called overdamping.

We just did
m=4, c=20, k=169, x_0=4, v_0=16
with equation
$4x'' + 20x' + 169x = 0.$

Then m=4, c=20, k=16, x_0=4, v_0=16 with equation
$4x'' + 20x' + 16x = 0$

Now we do m=4, c=20, k=25, x_0=4, v_0=16
with equation
$4x'' + 20x' + 25x = 0$

$(2r+5)^2 = 0$, $r = -\frac{5}{2}$ twice

General solution

$$x(t) = A e^{-5/2 t} + B t e^{-5/2 t}$$

$A = 4$ $B = 16$

This provides minimal damping without oscillation. The roots are real and repeated. This is critical damping.

For motion $x(t) = 3\cos(5t) - 4\sin(5t)$

A. what is the period? $2\pi/5$

B. what is the frequency? $5/2\pi$

C. what is the amplitude? 5

a. 5

b. $2\pi/5$

c. $5/2\pi$

d. $1/5$

e. 4

Summary:

1. Under-damped: the characteristic equation has

a. distinct real roots
b. repeated real roots
c. non-real complex conjugate roots ✓

2. Over-damped: the characteristic equation has

a. distinct real roots ✓
b. repeated real roots
c. non-real complex conjugate roots

3. Critically damped: the characteristic equation has

a. distinct real roots
b. repeated real roots ✓
c. non-real complex conjugate roots